MAXIMA & MINIMA

STRICTLY INCREASING FUNCTION :

A function f(x) is said to be a strictly increasing function on (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all } x_1, x_2 \in (a,b)$$

Stricly Decreasing Function : A function f(x) is said to be a strictly decreasing function on (a, b) if

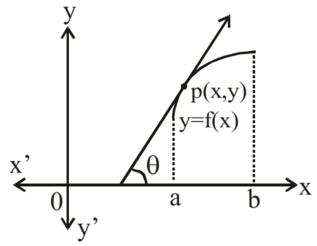
$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in (a,b)$$

By an increasing or a decreasing function we shall mean a strictly increasing or a strictly decreasing function.

Monotonic Function : A function f(x) is said to be monotic on an interval (a, b). it is either increasing or decreasing on (a, b)

Definiton : A function f(x) is said to be increasing on [a, b] if it is increasing on (a, b) and it is also increasing at x = a and x = b.

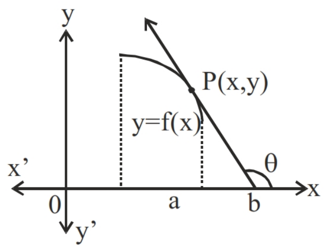
Necessary Condition : We observe that if f(x) is an increasing function on(a, b) then tangent at every point on the curve y = f(x) makes an acute angle θ with the positive direction of x-axis.



$$\therefore \tan\theta > 0 \Rightarrow \frac{dy}{dx} > 0 \text{ or } f'(x) > 0 \text{ for all } a \in (a,b)$$

If f(x) is a decreasing function on (a, b), then tangent at every point on the curve y = f(x) makes an obtuse angle θ with the positive direction of x-axis.

$$\therefore \tan\theta < 0 \Rightarrow \frac{dy}{dx} < 0 \text{ or } f'(x) < 0 \text{ for all } x \in (a,b)$$



SUFFICIENT CONDITION

THEOREM : Let f be a differentiable real function defined on an open interval (a,b)

(i) If $f'(x) > 0$ for all $x \in (a,b)$ then f(x) is increasing on (a, b)

(ii) If $f'(x) < 0$ for all $x \in (a,b)$, then f(x) is decreasing on (a,b).

Properties of Monotonic Function :

(i) If f(x) is strictly increasing function on an interval [a, b], then f^{-1} exists and it is also a strictly increasing function.

(ii) If f(x) is strictly increasing function on an interval [a, b] such that it is continuous, then f^{-1} is continuous on [f(a), f(b)].

(iii) If f(x) is continuus on [a,b] such that $f'(c) \geq 0$ $(f'(c) > 0)$ for each $c \in (a,b)$, then f(x) is monotonically (strictly) increasing function on [a,b]

(iv) If f(x) and g(x) are monotonically (or stricly) increasing (or decreasing) functions on [a, b], then gof (x) is a monotonically (or strictly) increasing function on [a,b]

(v) If one of the two functions f(x) and g(x) is strictly (or monotonically) increasing and other a strictly (monotonically) increasing and other a strictly (monotonically) decreasing, then gof(x) is strictly (monotonically) decreasing on [a, b].

MAXIMA AND MINIMA

Let f(x) be a function with domain $D \subset R$. Then f(x) is said to attain the maximum value at a point $a \in D$ if

$$f(x) \leq f(a) \text{ for all } x \in D$$

In such a case, a is called the point of maximum and f(a) is known as the maximum value or the greatest value.

Local Maximum : A function f(x) is said to attain a local maximum at x = a if there exists a neighbourhood . $(a-\delta, a+\delta)$ of a such that

$$f(x) < f(a) \text{ for all } x \in (a-\delta, a+\delta), x \neq a$$

or $$f(x) - f(a) < 0 \text{ for all } x \in (a-\delta, a+\delta),\ x \neq a$$

In such a case f(a) is called the local maximum value of f(x) at x = a.

Local Minimum : A function f(x) is said to attain a local minimum at x = a if there exists a neighbourhood $(a-\delta, a+\delta)$ of a such that f(x) > f(a) for all $x \in (a-\delta, a+\delta), x \neq a$

or $$f(x) - f(a) > 0 \text{ for all } x \in (a-\delta, a+\delta), x \neq a$$

The value of the function at x = a i.e. f(a) is called the local minimum value of f(x) at x = a

Theorem : A necessary condition for f(a) to be an extreme value of a function f(x) is that $f'(a) = 0$ in case it exists. A function may however attain an extreme value at a point without being derivable thereat. For example, the function $f(x) = |x|$ attains the minimum value at the origin even though it isnot derivable at $x = 0$.

Remark : Above condition is only a necessary condition for the point. x = a to be an extreme point. It is not sufficient i.e. $f'(a)$ does not necessarily imply that x = a is an extreme point. For example for the function $f(x) = x^3$, $f'(0) = 0$ but at x = 0 the function does not attain an extreme value.

Remark : The value of x for which $f'(x) = 0$ are called stationary values or critical values of x and the corresponding values of f(x) are called stationary or turning values of f(x).

Theorem : (First derivative test for local maximum and minima) Let f(x) be a function differentiable at x = a. Then,

(A) $x = a$ is a point of local maximum of f(x), if

(i) $f'(a) = 0$ and

(ii) $f'(x)$ changes sign from positive to negative as x passes through a i.e. $f'(x) > 0$ at every point in the left nbd $(a - \delta, a)$ and $f'(x) < 0$ at every point in the right nbd $(a, a + \delta)$ of a.

(B) $x = a$ is a point of local minimum of f(x), if

(i) $f'(a) = 0$ and

(ii) $f'(x)$ changes sign from negative to positive as x passes through a i.e. $f'(x) < 0$ at every point in the left nbd $(a - \delta, a)$ of a and $f'(x) > 0$ at every point in the right nbd $(a, a + \delta)$ of a.

(C) If $f'(a) = 0$ but $f'(x)$ does not change sign, that is $f'(a)$ has the same sign in the complete nbd of a , then a is neither a point of local maximum nor a point of local minimum.

Theorem : (Higher order derivative test). Let f be a differentiable function on an interval I and let c be an interior point of I such that

(i) $f'(c) = f''(c) = f'''(c) = \ldots = f^{n-1}(c) = 0$ and

(ii) $f^n(c)$ exists and is non-zero

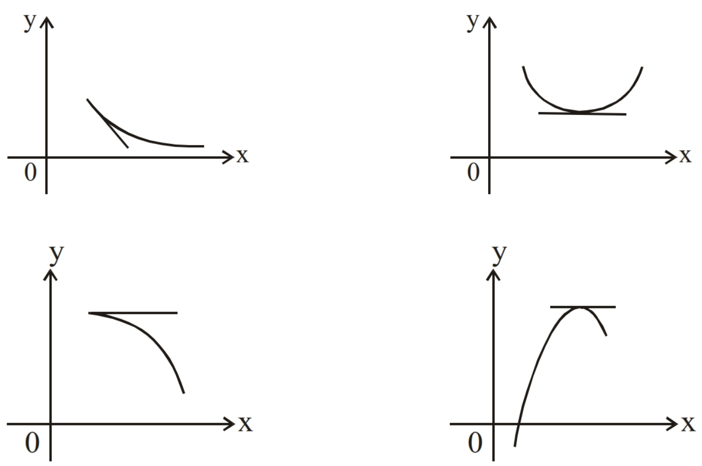
Then,

(a) If n is even and $f^n(c) < 0 \Rightarrow x = c$ is a point of local maximum.

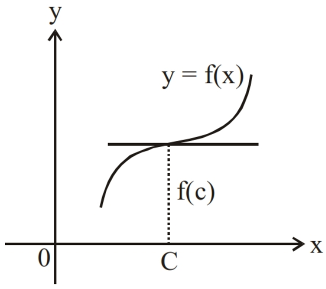
(b) If n is even and $f^n(c) > 0 \Rightarrow x = c$ is a point of local minimum

(c) If n is odd $\Rightarrow x = c$ is neither a point of local maximum nor a point of local minimum.

Point of inflection : An arc of a curve y = f(x) is called concave upward if, at each of its points, the arc lies above the tangent at the point. An arc of a curve y = f(x) is called concave downward if, at each of its points, the arc lies below the tangent at the point.



Definition : A point of inflection is a point at which a curve is changing concave upward to concave downward, or vice-versa.



A curve y = f(x) has one of its points x = c as an inflection point

If $f''(c) = 0$ or is not defined and

If $f''(x)$ changes sign as x increases through x = c.

The later condition may be replaced by $f'''(c) \neq 0$ when $f'''(c)$ exists.

Thus, $x = c$ is a point of inflection if $f''(c) = 0$ and $f'''(c) \neq 0$.

Critical point : A point $x = \alpha$ is a critical point of a function f(x) if $f'(\alpha) = 0$ or $f'(\alpha)$ does not exist.